Additional information about the analysis of numerical solution on nonlinear model:

The primary winding of the toroidal iron-cored two-windings transformer used as a nonlinear coil was modelled as nonlinear inductance $L_i(\phi)$, representing the magnetisation effect, in parallel with a nonlinear resistance $i_R(u_R)$, representing the core losses [E1]. Actual measurement values to fit with a piece-wise linear representation as shown in Fig. E1 have been used for the modelling of transformer magnetisation characteristics $L_i(\phi)$ and transformer core losses $i_R(u_R)$.

The time domain behaviour of the ferroresonant circuit in Fig. 3 can be described by a mathematical model composed of the second-order differential equation which is solved numerically:

$$\frac{d^2 \phi}{dt^2} + \frac{1}{C} L_i \left( \frac{d \phi}{dt} \right) + \frac{1}{C} i_R(\phi) = \frac{du}{dt}$$  \hspace{1cm} (E1)

The numerical simulation was carried out in the same way as the experimental investigation described in the section II, that is, the RMS source voltage value was increased from 0.4V to 60V with a step of 0.4V. Following this the value was reversed to its initial value of 0.4V using the same step size.

Simulation results shown in Fig. E2 indicate some resemblance to those of measurements with respect to the impact of number of coil turns and capacitance on the initiation of steady-state responses. However, appearance of sub-harmonic steady-state responses is somewhat curtailed whereas no more complex behavior (MCB) exists. The reason behind this discrepancy is the increased introduction of losses in the model and it is possible to initiate these steady-state responses by way of reducing the losses introduced to the ferroresonant circuit model as reported in [E2], [E3], [E4].

One way of reducing the losses is to revert to a linear resistance. For the next set of simulations the resistance in the circuit was modified to a linear one with values given in Table E-I. These values are based on the linear part of the corresponding resistance characteristics in Fig. E1. Aside from the initiation of sub-harmonic and chaotic steady-states, this linearization of losses introduces simplification to the model. In comparison to (E1) the circuit model can now be described with the following equation:

$$\frac{d^2 \phi}{dt^2} + \frac{1}{RC} \frac{d \phi}{dt} + \frac{1}{C} i_L(\phi) = \frac{du}{dt}$$  \hspace{1cm} (E2)

The results of simulation carried out using linear resistance are shown in Fig. E2. These exhibit similar properties to measurement results regarding the impact of the number of coil turns and capacitance on the initiation of steady-state responses. Contrary to simulation results obtained using nonlinear resistance, the sub-harmonic steady-states that are being searched (P2 and P3) and more complex behavior appear in a greater extent suggesting that linearization of losses is more accurate in terms of prediction of all characteristic steady-state responses. However it should be noted that the sub-harmonic and more complex behavior appear at lower RMS source voltage values compared to measurement results. Furthermore, contrary to measurements, occurrence of simpler steady-states that are being searched is observed inside the area of more complex behavior.
Additional information about the stability analysis

Floquet multipliers have been used as a test of the stability limit cycle solution. According to [E5], Floquet exponents/multipliers can be interpreted in the same way as eigenvalues are in models with constant coefficients in continuous/discrete time, respectively; they represent the growth rate of different perturbations averaged over a cycle. Floquet exponents are rates with units time$^{-1}$, and Floquet multipliers are dimensionless numbers that give the period-to-period increase/decrease of the perturbation. In general, absolute value of Floquet multipliers is indicator of stability, i.e. a multiplier higher than one indicates a change of steady-state type. Thereby, particular values of multipliers defines in detail which steady-state type occurred, e.g. if one of the real parts of multipliers passes through −1, it is an indicator of a period-doubling bifurcation of the limit cycle.

In order to determine Floquet multipliers first we located the limit cycle solution numerically and one point on the limit cycle, as well as its period $T$, such that $x(t + T) = x(t) = x_1$, $y(t + T) = y(t) = y_1$. Then we added small perturbation $\Delta x$ to $x_1$ and calculate the variable values $x_2$ and $y_2$ after one period $T$. Then the same procedure is carried out for variable $y_1$ as well ($\Delta y$ added to $y_1$) to obtain values $x_3$ and $y_3$. Finally, the Floquet multipliers are calculated as eigenvalues of matrix defined as:

$$J = \frac{x_2 - x_1}{\Delta x} \quad \frac{x_3 - x_1}{\Delta y}$$

Absolute values of Floquet multipliers shown on Fig. E4 are calculated in this way for parameter values that are corresponding to steady-states shown on Fig. 4c. Thereby, $\Delta x$ and $\Delta y$ has been chosen to be equal to 0.00001.

Comparison of Fig. E4 and steady-states shown on Fig. 4c) reveals that the change of steady-state type is in general indicated by an absolute value of Floquet multiplier that is higher than one. However, some of steady-state types are followed by a significant change of absolute value of Floquet multiplier that does not reach value of one. The reason for this discrepancy could lie in used numerical method of calculation of Floquet multipliers or it could be inherent property of flux reflection model. Both assumptions will be investigated in future research.
FIGURES AND TABLES:

**Fig. E1.** Characteristics of toroidal iron core used in the model:

- **a)** magnetisation characteristics \( i_L (\phi) \)
- **b)** core losses \( i_R (u_R) \)

**Fig. E2.** Simulated response maps and bifurcation diagram utilising nonlinear inductance and resistance

- **a)** \( U_{N1} = 24 \text{ V} \);
- **b)** \( U_{N1} = 27 \text{ V} \);
- **c)** \( U_{N2} = 30 \text{ V} \);
- **d)** \( U_{N3} = 33 \text{ V} \);
- **e)** \( U_{N4} = 36 \text{ V} \);
- **f)** Bifurcation diagram for \( u_C \) with \( U_{N1} = 30 \text{ V} \), \( C = 20 \, \mu\text{F} \) as marked on c)
Fig. E3. Simulated response maps and bifurcation diagram employing nonlinear inductance and linear resistance

- a) $U_{N1} = 24 \text{V}$
- b) $U_{N1} = 27 \text{V}$
- c) $U_{N1} = 30 \text{V}$
- d) $U_{N1} = 33 \text{V}$
- e) $U_{N1} = 36 \text{V}$
- f) Bifurcation diagram for $u_C$ with $U_{N1} = 30 \text{V}$, $C = 20 \mu F$ as marked on c)

<table>
<thead>
<tr>
<th>Coil nominal voltage $U_n$</th>
<th>$U_{N1} = 24 \text{V}$</th>
<th>$U_{N1} = 27 \text{V}$</th>
<th>$U_{N1} = 30 \text{V}$</th>
<th>$U_{N1} = 33 \text{V}$</th>
<th>$U_{N1} = 36 \text{V}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resistance $R$</td>
<td>568.4 $\Omega$</td>
<td>757.9 $\Omega$</td>
<td>936.2 $\Omega$</td>
<td>1136.8 $\Omega$</td>
<td>1326.3 $\Omega$</td>
</tr>
</tbody>
</table>
a) $C = 10 \ \mu F$

b) $C = 15 \ \mu F$
c) $C = 20 \ \mu F$

d) $C = 25 \ \mu F$
Fig. E4. Absolute values of Floquet multipliers obtained for parameter values that are corresponding to steady-states shown on Fig. 4c

REFERENCES: